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Thermodynamic criteria governing the stability of fluctuating paths in the limit of small thermal fluctuations: Critical paths and temporal bifurcations†

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Abstract. Thermodynamic criteria for the stability of the most probable path for a fluctuation are derived in the limit of small thermal fluctuations. The positive definiteness of the second Frechet differential of the Onsager–Machlup functional insures that the asymptotic probability distribution will be Gaussian about that extremal path at which the Onsager–Machlup functional is a proper minimum. The vanishing of one of the eigenvalues of the second Frechet differential of the Onsager–Machlup functional indicates the presence of a critical path. The lowest-order term in the extended form of the Taylor series expansion of the Onsager–Machlup functional determines the stability characteristics of the critical path. The cubic case is examined in detail.

1. Introduction and summary

There appears to be a rather profound analogy between equilibrium and non-equilibrium thermodynamic critical phenomena. Equilibrium thermodynamic stability criteria are couched in the convexity of thermodynamic potentials. These potentials govern the stability of the 'most probable' or equilibrium state. In the case of second-order phase transitions (see, for example, Landau and Lifshitz 1969), the critical point is determined by the vanishing of the second variation of the thermodynamic potential. The question of stability in the critical region rests with the higher-order terms in the Taylor series expansion of the thermodynamic potential about the equilibrium state. In order to insure stability in the critical region, the third-order terms vanish and the fourth-order terms must be positive definite. The vanishing of the third-order terms attests to the fact that the thermodynamic potential cannot be a function of the sign of the fluctuation. Beyond the critical point, the initial state becomes unstable and two new states appear which are symmetrically arranged about the unstable state. These states are determined from the stationary condition of a potential which is a fourth-order polynomial in the 'order' parameter.

Non-equilibrium phase transitions can manifest themselves as bifurcations in the branch solution of the kinetic equations that would reduce to thermodynamic equilibrium in the absence of the non-equilibrium constraint (Lavenda 1970). The unstable transition is determined by a competition between the opposing effects of random thermal fluctuations and collective, deterministic motion. Once beyond the transition

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point, we shall see that the two opposing factors swap roles. In the limit of small thermal fluctuations, we would expect the asymptotics to reflect the statistical independence of the process, analogous to a central limit theorem and the law of large numbers (cf expression (21) and following discussion), were it not for the predominance of the collective motion. The breakdown in the central limit theorem would be significant of large correlations in a certain mode. The amplitude of this mode grows to a finite value, which is the order parameter of the phase transition. This mode has a zero frequency and at zero frequency, random thermal fluctuations are a dominant noise source. Such forms of non-equilibrium bifurcation phenomena are well known; for example, they occur in the laser start-up (Scully and DiGiorgio 1970) where the unstable transition leads to a state in which there is an emission of nearly monochromatic, intense radiation, in the DC Josephson junction (Ambegaokar and Halpern 1969), where at temperatures sufficiently close to the transition temperature, thermal fluctuations can disrupt the coupling of phases of the order parameters of two superconductors separated by a thin insulating barrier, and in the transition from a laminar to a turbulent state in hydrodynamics (Landau 1937, Ruelle and Takens 1971).

Thermodynamic criteria governing non-equilibrium statistical processes in the limit of small thermal fluctuations have been derived in Lavenda and Santamato (1982). The asymptotic results were obtained from estimates involving probabilities and their densities. Other types of asymptotic results pertain to averages with respect to the Wiener integral. Rigorous results on the asymptotic expansion of the Wiener integral have been obtained by Schilder (1966) by involving Laplace's method. What Schilder showed was that a parallel translation of the probability measure associated with a diffusion process with a finite drift transforms it into a new probability measure such that the most probable trajectory of the diffusion process with respect to the new measure turns out to be the extremal of a certain functional in the case of a small characteristic parameter. The new probability measure is asymptotically Gaussian about the extremal. The application of Laplace's method for Gaussian integrals has further been developed by Ellis and Rosen (1982a). Ellis and Rosen (1980, 1982b) generalised the Laplace method to simple degenerate and coalescing minima. The stationary phase analogue of Laplace's method for the quantum mechanical propagator has been developed by Schulman (1975, 1981).

In this paper, we show that the Onsager–Machlup (OM) functional (Lavenda and Santamato 1982) governs the stability non-equilibrium thermodynamic statistical processes in the limit of small thermal fluctuations. This functional determines the stability of the most probable path of a fluctuation in an analogous manner that an equilibrium thermodynamic function determines the stability of the equilibrium or most probable state. On the basis of the validity of the Laplace method for Wiener integrals (Schilder 1966, Ellis and Rosen 1982a), we show that the stability of the most probable path of a non-equilibrium fluctuation is governed by the sign of the second Frechet differential of the OM functional. The most probable path is stable when the second Frechet differential is positive definite. 'Critical' paths occur when the second Frechet differential vanishes. This corresponds to a coalescence and disappearance of most probable paths in analogy to the formation of a focal or conjugate point on a caustic surface (Schulman 1975, 1981). The stability characteristics of the critical paths are determined by the lowest-order non-vanishing Frechet differential of the OM functional in an analogous way that the lowest-order term in the Taylor series expansion of a thermodynamic potential determines the stability properties of new thermodynamic states that emerge beyond the critical point. Depending on this order, we obtain polynomial expressions, in the unstable mode about the critical path, which give rise to cuspoid catastrophies (see, for example, Poston and Stewart 1978). However, unlike equilibrium second-order phase transitions, we cannot invoke symmetry arguments for why odd powers in an extended form of a Taylor series expansion of the OM functional should vanish. For example, if there is a non-vanishing third-order Frechet differential of the OM functional then a so-called fold results. Therefore, there will be a certain polynomial which appears in an integral with a small parameter, which we identify with Boltzmann's constant (Lavenda and Santamato 1982), that will govern the form of the critical phenomenon.

2. Transformation of functional integrals

Integrals over the Wiener measure can be transformed into integrals over new probability measures which are absolutely continuous with respect to the Wiener measure (Girsanov 1960). Although this property enables us to derive an expression for the probability distribution of paths of non-equilibrium statistical thermodynamic processes (Lavenda and Santamato 1981), we prefer to use the standard transformation formulae of Gel'fand and Yaglom (1960) since they provide for a continuity in approach and connect a Jacobian of a generally nonlinear transformation with the correction term in Itô's stochastic calculus.

A particle moving under the influence of Brownian motion in $\mathbb{R}^r$ is described by the stochastic differential equation:

$$d\varphi(t) = (2k)^{1/2}\, dW(t), \qquad \varphi(0) = 0, \tag{1}$$

where $W(t)$ is a standard Brownian motion whose intensity parameter, k, is identified as Boltzmann's constant (Lavenda and Santamato 1982). Denote by

$$E_\varphi^W\{F(\sqrt{k}\varphi)\} = \int_c F(\sqrt{k}\varphi)\, dP(\varphi), \tag{2}$$

the integral of a functional $F(\varphi)$ belonging to the class of all bounded and continuous functionals over a function space $C[0, T]$ of continuous functions in the interval $[0, T]$ such that $\varphi(0) = 0$. This initial condition is no restriction and is used for simplicity: any other initial condition will simply cause a shift in the arguments in expression (3) (Gel'fand and Yaglom 1960). $P(\varphi)$ denotes the Wiener measure, namely,

$$dP(\varphi) = (4\pi kT)^{-1/2} \exp(-\varphi^2/4kT)\, d\varphi. \tag{3}$$

The scaling in (2) has been introduced to suggest an analogy between the asymptotic 6behaviour of $\{\varphi^k\}$ and the law of large numbers (cf (32)).

We are interested in how this measure transforms under the general *nonlinear* transformation:

$$\varphi(T) \to \tilde{\varphi}(T) = \varphi(T) - \int_0^T b(\varphi)\, dt, \tag{4}$$

where the drift, b, is assumed to be bounded and continuous. By the well known transformation properties of Wiener integrals (Schilder 1966), we get

$$E_{\tilde{\varphi}}^W\{F(\sqrt{k}\tilde{\varphi})\} = E_\varphi^W\{F(\sqrt{k}\varphi)J(\sqrt{k}\varphi) \exp[\Lambda(\sqrt{k}\varphi)/2k]\}, \tag{5}$$

where

$$\Lambda(\varphi) = \int_0^T b(\varphi)\,\mathrm{d}\varphi - \int_0^T V(\varphi)\,\mathrm{d}t \tag{6}$$

and $V(\varphi)$ is the so-called generating function (Landau and Lifshitz 1969):

$$V(\varphi) := \tfrac{1}{2}b^2(\varphi). \tag{7}$$

The role of the Jacobian is taken by the linear part of transformation (4). Observing that one half of Volterra's kernel on the diagonal (Gel'fand and Yaglom 1960) is $-\frac{1}{2}b'(\varphi)$, Fredholm's determinant is equal to:

$$J(\varphi) = \exp\left(-\tfrac{1}{2}\int_0^T b'(\varphi)\,\mathrm{d}t\right). \tag{8}$$

Since the transformation is nonlinear, the Jacobian is clearly a functional of the path $\varphi = \varphi(t)$, and consequently it cannot be brought out from under the expectation sign in (5). On the contrary, if we had employed Girsanov's theorem (Girsanov 1960, Lavenda and Santamato 1981), the first integral in (6) would have turned out to be an Itô (I)-stochastic integral and its conversion to the symmetric Fisk–Stratonovich (S)-stochastic integral (Stratonovich 1968):

$$\begin{aligned}(I)-\int_0^T b(\varphi)\,\mathrm{d}\varphi &= (S)-\int_0^T b(\varphi)\,\mathrm{d}\varphi - \tfrac{1}{2}\int_0^T b'(\varphi)(\mathrm{d}\varphi)^2 \\ &= (S)-\int_0^T b(\varphi)\,\mathrm{d}\varphi - k\int_0^T b'(\varphi)\,\mathrm{d}t \end{aligned} \tag{9}$$

would have generated automatically the Jacobian of the transformation. Note that $(\mathrm{d}\varphi)^2$ has been replaced by its Brownian motion expectation value, $2k\,\mathrm{d}t$. Since the Fisk–Stratonovich integral enjoys all the properties of an ordinary integral (Stratonovich 1968), the first integral on the right-hand side of (9) is identified with the first integral on the right-hand side of (6). The second integral in (9) is the uncertainty in the specification of a Brownian path (Lavenda and Santamato 1979); alternatively J^2 can be interpreted as the density of paths satisfying the continuity equation (Gel'fand and Yaglom 1960):

$$\partial_t J^2 + \partial_\varphi(bJ^2) = 0. \tag{10}$$

The asymptotic behaviour of the functional average, in the 'thermodynamic' limit as $k\downarrow 0$ (Lavenda and Santamato 1982), can be obtained by Laplace's method (Schilder 1966, Ellis and Rosen 1982a). Suppose that the OM functional

$$\Theta(\varphi) = \tfrac{1}{2}\int_0^T \dot{\varphi}^2\,\mathrm{d}t - \Lambda(\varphi) \tag{11}$$

has a proper minimum on $C[0, T]$. Call this minimising function $\varphi^*(t)$. Let the functional average (5) undergo a parallel translation,

$$\varphi(t) \to h(t) = \varphi(t) - \varphi^*(t)/\sqrt{k}, \tag{12}$$

in the functional space $C[0, T]$ with endpoints

$$h(0) = h(T) = 0. \tag{13}$$

Denoting the functional average (5) by $I(k)$ we have:

$$I(k)=\exp\left[-\left(\frac{1}{4k}\right)\int_0^T \dot{\varphi}^{*2}\,\mathrm{d}t\right]E_h^W\left\{F(\varphi^*+\sqrt{k}h)J(\varphi^*+\sqrt{k}h)\right.$$
$$\left.\times\exp\left[-\left(\frac{1}{2k}\right)\left(\sqrt{k}\int_0^T \dot{\varphi}^*\,\mathrm{d}h-\Lambda(\varphi^*+\sqrt{k}h)\right)\right]\right\}. \tag{14}$$

If the functional $\Lambda(\varphi)$ possesses at least two Frechet differentials in the neighbourhood of the minimising function then by an extended form of Taylor's theorem we have

$$\Lambda(\sqrt{k}\varphi)=\Lambda(\varphi^*)+\sqrt{k}\,\mathrm{d}\Lambda(\varphi^*)h+(k/2!)\,\mathrm{d}^2\Lambda(\varphi^*)h^2+\ldots. \tag{15}$$

Upon introducing (15) into expression (14) we obtain:

$$I(k)=\exp[-(\tfrac{1}{2}k)\Theta(\varphi^*)]E_h^W\{F(\varphi^*+\sqrt{k}h)J(\varphi^*+\sqrt{k}h)\exp[\tfrac{1}{4}\,\mathrm{d}^2\Lambda(\varphi^*)h^2]\}, \tag{16}$$

where $E_h^W\{\cdot\}$ denotes integration with respect to the 'excess' Wiener measure:

$$\mathrm{d}P(h)=(4\pi T)^{-1/2}\exp\left(-\tfrac{1}{4}\int_0^T[\dot{h}(t)]^2\,\mathrm{d}t\right)\mathrm{d}h. \tag{17}$$

Because

$$\Theta(\varphi^*)=\tfrac{1}{2}\int_0^T\dot{\varphi}^{*2}\,\mathrm{d}t-\Lambda(\varphi^*)$$

is the proper minimum implying that the first Frechet differential of the OM functional vanishes along $\varphi^*(t)$, namely

$$\int_0^T h\ddot{\varphi}^*\,\mathrm{d}t+\mathrm{d}\Lambda(\varphi^*)h=0 \tag{18}$$

which is equivalent to the Euler–Lagrange equation:

$$\ddot{\varphi}^*-V'(\varphi^*)=0. \tag{19}$$

In order to obtain (19), we have performed an integration by parts and used the homogeneous boundary conditions (13). Finally, taking the thermodynamic limit, (16) becomes

$$\lim_{k\downarrow 0}I(k)=F(\varphi^*)J(\varphi^*)\exp\left[-\left(\frac{1}{2k}\right)\Theta(\varphi^*)\right]E_h^W\{\exp[\tfrac{1}{4}\,\mathrm{d}^2\Lambda(\varphi^*)h^2]\}. \tag{20}$$

This expression can also be written in the form:

$$\lim_{k\downarrow 0}I(k)=F(\varphi^*)J(\varphi^*)\exp\left[-\left(\frac{1}{2k}\right)\Theta(\varphi^*)\right]E_h^W\left[\exp\left(-\tfrac{1}{4}\int_0^T V''(\varphi^*)h^2\,\mathrm{d}t\right)\right], \tag{21}$$

reminiscent of a Feynman–Kac formula.

What we have accomplished is to transform the original measure for φ into a new measure for h such that the most probable path of φ with respect to the new measure is φ^* for small k. And with respect to the new probability measure, $h=\varphi-\varphi^*/\sqrt{k}$ is Gaussian provided the OM functional has a proper minimum at φ^*. This can be thought of as a law of large numbers or a central limit theorem for $\{\varphi^k\}$ (Ellis and Rosen 1982b) in the asymptotic limit as $k\downarrow 0$. The statistical independence of $\{\varphi^k\}$ will be related to a stability criterion in the next section.

3. Stability of fluctuating paths in the small k limit

If the OM functional is to have a proper minimum at φ^* then

$$d^2\Theta(\varphi^*)h^2 > 0. \tag{22}$$

Now the expectation in expression (20) can be written as

$$\int_c \exp[\tfrac{1}{4} d^2\Lambda(\varphi^*)h^2]\, dP(h) = \int_c \exp[-\tfrac{1}{4} d^2\Theta(\varphi^*)h^2]\, dh/(4\pi T)^{1/2} \tag{23}$$

where $P(h)$ is the excess Wiener measure defined in (17). Expression (23) clearly shows that the minimum property (22) is a stability criterion for the most probable path of a fluctuation in the thermodynamic limit as $k\downarrow 0$. In this section, we analyse this criterion in greater detail.

The question of the positive definiteness of the second Frechet differential of the OM functional can be answered in terms of a variational problem (Schulman 1981). What we want to determine is whether there is any function ψ for which $d^2\Theta(\varphi^*)h^2$ vanishes. The function $\psi(t)$ must satisfy the same homogeneous boundary conditions as $h(t)$ (cf (13)). In addition, it must be normalised on the interval [0, T], namely,

$$\int_0^T (\psi(t))^2\, dt = 1, \tag{24}$$

since the homogeneous boundary conditions set no scale for the variational problem. The normalisation condition acts as a constraint on the variational problem

$$d^2\Theta(\varphi^*)\psi^2 = \int_0^T (\dot{\psi}^2(t) + V''(\varphi^*)\psi^2(t))\, dt = \min. \tag{25}$$

The second Frechet differential of the OM functional (25) resembles an action whose corresponding Lagrangian is

$$L(\psi) := \dot{\psi}^2 + V''(\varphi^*)\psi^2. \tag{26}$$

Handling the normalisation constraint by the method of Lagrange multipliers, the unconstrained variational principle corresponding to (25) yields the extremum condition

$$\ddot{\psi} - V''(\varphi^*)\psi + \lambda\psi = 0, \tag{27}$$

where λ is the Lagrange multiplier. Equation (27) is the well known Jacobi equation. Multiplying the Jacobi equation by $\psi(t)$, integrating over the interval [0, T], and performing an integration by parts show that the Lagrange multiplier is the minimum value of the OM functional

$$d^2\Theta(\varphi^*)\psi^2 = \lambda. \tag{28}$$

The Jacobi equation (27) together with the homogeneous boundary conditions constitute a Sturm–Liouville problem. We therefore know that the eigenfunctions of (27) constitute an infinitely denumerable sequence whose corresponding eigenvalues are ordered such that $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$. Although only the eigenfunction corresponding to the smallest eigenvalue minimises $d^2\Theta(\varphi^*)h^2$, the other eigenfunctions do make it stationary (Schulman 1981). Since we are working in function space, where each eigenfunction or mode represents a 'direction', we will want to keep the space finite.

This means that we will truncate the infinite series at some finite value N and set

$$h(t)=\sum_{j=1}^{N} a_j\psi_j(t) \tag{29}$$

where the a_j are expansion coefficients. In the unscaled case, the error for each $a_j, j=1,\ldots,N$ will go to zero with k (Schulman 1981). Hence, the second Frechet differential of the OM functional can be written in the canonical form

$$\mathrm{d}^2\Theta(\varphi^*)h^2=\sum_{j=1}^{N}\lambda_j a_j^2. \tag{30}$$

The stability question can thus be answered by showing that the smallest eigenvalue is positive. Introducing (30) into (23) and with the change of variables to integration over the modes we obtain:

$$\begin{aligned}E_h^W\{\exp[\tfrac14\,\mathrm{d}^2\Lambda(\varphi^*)h^2]\} &= \text{constant}\times\int\prod_{j=1}^{N}\mathrm{d}a_j\exp\left(-\tfrac14\sum_{j=1}^{N}\lambda_j a_j^2\right)\\ &= \text{constant}\left(\prod_{j=1}^{N}\lambda_j\right)^{-1/2}.\end{aligned} \tag{31}$$

where the constant contains the Jacobian of the transformation from coordinates to modes. From the Sturm–Liouville theory, we know that (31) will turn out to be finite only when $\lambda_1>0$.

Formula (31) will break down in the region near and at focal or conjugate points where at least one eigenvalue of the second Frechet differential of the OM functional goes to zero. We will assume, in the next section, that $\varphi^*(t)$ is simply degenerate. The path $\varphi^*(t)$ will be referred to as a critical path whose stability properties will now be analysed.

4. Critical paths and temporal bifurcations

Suppose that the critical point is a simple focal point with a single eigenvalue vanishing at that point. In the critical region, the functional average (20) is most singular and we restrict our attention to this domain.

In the non-critical case, the stability criterion (22) holds and asymptotically

$$(\varphi-\varphi^*/\sqrt{k})\overset{\mathscr{D}}{\rightarrow}P(h) \tag{32}$$

where $\overset{\mathscr{D}}{\rightarrow}$ denotes convergence in distribution. It is apparent that (32) suggests an analogy with the central limit theorem for a large number of independent events. Now in the critical region, the path stability criterion (22) is violated and we can no longer expect (32) to hold. This implies the presence of extremely large statistical correlations in the critical region. In the critical region (32) must be replaced by:

$$k^{(1/2-1/\nu)}(\varphi-\varphi^*/\sqrt{k})\rightarrow Q_\nu(h) \tag{33}$$

where $Q_\nu(h)$ is a non-Gaussian probability measure on $\mathbb{R}^1$ which is determined by the critical exponent $(1/\nu)$ (Ellis and Rosen 1982b). The integer ν is determined by the lowest non-vanishing Frechet derivative of the functional Λ, i.e., $(1/\nu)\in(0,\frac12)$.

In the absence of scaling the a_1 mode is of $O(k^{1/2})$ for normal fluctuations. However, in the critical region it is of $O(k^{1/\nu})$ which can be seen from the following simple example. Suppose that there is a finite third-order Frechet derivative of Λ. We would then be led to consider integrals of the type

$$Y_m(k)=\int y^m \exp[-(1-k)y^3]\,\mathrm{d}y. \tag{34}$$

Assuming that the limits of integration are unimportant, we would obtain a scaling of the form

$$Y_m(k)=k^{m/3}k^{1/3}Y_m(1), \tag{35}$$

showing that each power of y gives a contribution of $O(k^{1/3})$. In this case the critical exponent, $(1/\nu)=\frac{1}{3}$ and we must rescale accordingly in order to obtain a correct weighting of the terms entering into the extended Taylor series expansion of the OM functional. Let us consider this case in greater detail.

One of the eigenvalues of $\mathrm{d}^2\Theta h^2$ vanishes along the most probable path. The eigenvalue is a function of the endpoints as well as the time interval T. Thus the endpoint $\varphi^*(T)$ is a conjugate point for the trajectory leaving $\varphi^*(0)$. The conjugate point lies, in the more general case of $\mathbb{R}^d$ where $d>1$, on a caustic surface and the disappearance of a classical path is a characteristic feature at a caustic (Schulman 1975). Hence, we will consider an expansion about the most probable path in which there is a small shift Δ from φ^* and fix it so that the parameter Δ determines the size of the critical region. In order to rescale in the critical region, we extract the fundamental from (29), which we denote by h_1, and weight it by a factor $k^{1/3}$. The remaining modes will be denoted by h' and are weighted by the usual $k^{1/2}$. Hence the 'Taylor series' expansion for the OM functional in the critical region is:

$$\begin{aligned}\Theta(\varphi^*+\Delta+\sqrt[3]{k}h_1+\sqrt{k}h')\\ &=\Delta\int_0^T V''(\varphi^*)(\sqrt[3]{k}h_1+\sqrt{k}h')\,\mathrm{d}t+(k/2)\sum_{j=2}^{N}\lambda_j a_j^2\\ &\quad+(k/3!)\int_0^T V'''(\varphi^*)h_1^3\,\mathrm{d}t\end{aligned} \tag{36}$$

to leading order in k. In the derivation of (36), we have used the fact that φ^* is the solution of the functional equation (Schilder 1966)

$$\Theta(\varphi^*)=0 \tag{37}$$

and consequently,

$$-\mathrm{d}\Theta(\varphi^*)\Delta=\mathrm{d}\Lambda(\varphi^*)\Delta=\Delta\int_0^T b'(\varphi^*)(\dot{\varphi}^*-b(\varphi^*))\,\mathrm{d}t=0.$$

The dominant contribution to the integral over the a_1-mode will come from the cubic and linear terms. In order that they be of the same order of magnitude, the parameter is chosen to be of $O(k^{2/3})$ and this fixes the size of the critical region (Schulman 1981). The remaining term in Δ is smaller by $k^{1/6}$ than the other terms and hence can be discarded. The expansion of the OM functional in the critical region, to leading order in k, can be written in the canonical form:

$$\left(\frac{1}{2k}\right)\Theta(\varphi^*+{}^3\sqrt{k}h_1+\sqrt{k}h')=c_1a_1+\tfrac{1}{4}\sum_{j=2}^{N}\lambda_j a_j^2+\tfrac{1}{3}c_3a_1^3 \tag{38}$$

where

$$c_1=\tfrac{1}{2}\delta\int_0^T V''(\varphi^*)\psi_1^2(t)\,\mathrm{d}t \tag{39}$$

$$c_3=\tfrac{1}{4}\int_0^T V'''(\varphi^*)(\psi_1(t))^3\,\mathrm{d}t \tag{40}$$

and δ denotes the scaled critical region $\Delta/k^{2/3}$.

Consequently, the functional integral (5) in the small k limit in the critical region takes the form (cf expression (20))

$$\lim_{k\downarrow 0} I(k)=\text{constant}\times J(\varphi^*)F(\varphi^*)\int\prod_{j=1}^{N}\mathrm{d}a_j\exp\left(-\tfrac{1}{4}\sum_{j=2}^{N}\lambda_j a_j^2-(\tfrac{1}{3}c_3a_1^3+c_1a_1)\right). \tag{41}$$

The integrals over modes a_2 to a_N give a factor proportional to $[\Pi_j' \lambda_j]^{-1/2}$ where the prime denotes that the a_1 mode has been excluded. We are therefore left with the task of evaluating the integral over the cubic polynomial

$$Q(q)=\int \mathrm{d}a_1\,\exp\{-c_3[\tfrac{1}{3}a_1^3+qa_1]\} \tag{42}$$

in

$$\lim_{k\downarrow 0} I(k)=\text{constant}\times J(\varphi^*)F(\varphi^*)\left(\prod_j{}'\lambda_j\right)^{-1/2}Q(q) \tag{43}$$

where $q:=(c_1/c_3)$. Observe that (42) is closely related to the Airy integral

$$\mathrm{Ai}(q)=\int \mathrm{d}b_1\,\exp\{\mathrm{i}[qb_1-\tfrac{1}{3}b_1^3]\}. \tag{44}$$

The nature of the solution depends on the sign of q. Now since $q\leqslant k^{-2/3}$, the thermodynamic limit $k\downarrow 0$ sends $|q|$ to infinity. The asymptotic forms of the Airy integral for large $|q|$ can be found by the saddle-point method. For $q\to\infty$, the saddle points occur on the real axis and we must go over them. The asymptotic form of the Airy integral is

$$\mathrm{Ai}(q)\sim q^{-1/4}\cos(\tfrac{2}{3}q^{3/2}+\pi/4) \qquad \text{as } q\to\infty. \tag{45}$$

In regard to our original integral, this would correspond to a pure imaginary a_1 mode which gives the dominant contribution:

$$Q(q)\sim[c_1c_3]^{-1/4}\cos(\tfrac{2}{3}c_1q^{1/2}+\pi/4) \qquad \text{as } q\to\infty. \tag{46}$$

For $q\to-\infty$, the saddle points of (44) occur on the imaginary axis and we pass through that saddle point that has a zero inclination of the path at the saddle point. The Airy integral is approximately given by

$$\mathrm{Ai}(q)\sim(-q)^{-1/4}\exp[-\tfrac{2}{3}(-q)^{3/2}] \qquad \text{as } q\to-\infty. \tag{47}$$

This gives a real a_1 mode and the functional integral decays exponentially as $\exp\{-|\delta|^{3/2}\}$ since

$$Q(q) \sim (-c_1 c_3)^{-1/4} \exp(-\tfrac{2}{3}|c_1|\sqrt{|q|}) \qquad \text{as } q \to -\infty. \tag{48}$$

The sign of q is determined by whether we are prior to or beyond the focal point, corresponding to a negative or positive δ, respectively. The exponential decay region corresponds to the 'shadow' and there, the functional average (41) has a small value. Alternatively, the region beyond the conjugate point is 'brightly illuminated' and there will be a far greater probability of finding the system since the value of (41) is much larger. The brightest region is found in the immediate vicinity of the conjugate point; there, the probability is greatest for finding the system and the functional integral is most singular, i.e. $Q(0)$ is of $O(k^{1/6})$. It is in this region that we can expect highly anomalous effects to occur. As the width of the critical region diminishes the individual saddle points coalesce at the origin into a monkey saddle.

In contrast to deterministic stability criteria which determine whether states are stable or unstable, the stochastic analysis presented here determines zones in which the system is likely or unlikely to be found in. From this we conclude that an effect of random thermal fluctuations is to undermine the deterministic stability criteria.

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